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Stability of Proportional-Plus-Derivative-Plus-Integral Control of Flexible Spacecraft

Peter C. Hughes*
University of Toronto, Downsview, Ontario, Canada
and
Tarek M. Abdel-Rahman†
Spar Aerospace Limited, Toronto, Ontario, Canada

The linear attitude control of flexible spacecraft is considered. The feedback law is of the proportional-plus-derivative-plus-integral class. The sensor and actuator dynamics are included, albeit in simple models. The structural flexibility model is unrestricted except for the usual assumption of small deflections. The principal result of the paper is that if the controller is unconditionally stable (with respect to gain), assuming the satellite to be rigid, then structural flexibility cannot destabilize it. This and other possibilities are illustrated by numerical examples.

I. Introduction

ATTITUDE control systems for spacecraft are becoming increasingly sophisticated. One aspect of this evolution is their capability of handling the dynamical implications of flexible spacecraft structures. Unfurlable antennas and deployable solar arrays are typical of these relatively flimsy appendages.

It is not the objective of this paper to report results for a specific controller or a specific satellite configuration. Attention is addressed, instead, to discovering those features of the situation that are likely to be found in common with linear controllers and a wide range of flexible appendages. A recent review of flexible spacecraft dynamics by Modi[1] underlines the breadth of these categories. Many forms of spacecraft attitude controllers have been suggested, and each of these would have a large number of design parameters. In a similar manner, the dynamics of spacecraft with flexible appendages can, in general, lead to complicated analyses (see, for example, Likins[2]), and the flexible appendages can again be expected to contribute many design parameters. Even relatively idealized models of the solar arrays (e.g., Refs. 3 and 4) require many geometrical, inertial, and structural parameters.

Both of the primary approaches to control systems analysis ("classical" and "modern") are applicable to the attitude control of flexible space vehicles. The classical approach has been used for most existing spacecraft, but modern control theory is increasingly being applied (Ref. 5, for example). The classical method often is used in conjunction with the assumption of "mode separability," where it is assumed that the natural frequencies of spacecraft vibration are well separated from each other and from the so-called "rigid-body" poles.[6-8] When this assumption is reasonable, the interaction between the controller and the dynamical system can be analyzed approximately by considering one mode at a time and suppressing the remaining ones. The simplest form of this process is to design the controller by applying a conventional classical control systems approach for the rigid-body poles alone, and then to check the effects of the higher modes, one at a time. The design frequently becomes an iterative process because a subsequent mode may deteriorate system performance or even lead to instability. As a final verification for an actual system, a simulation should be run that incorporates all modes simultaneously.

To avoid such an iterative process, it is desirable to have a stability criterion that enables one to select the controller gains based on rigid-body poles but that does not jeopardize the stability of most flexible space vehicles. This criterion is derived in Sec. V. It is particularly useful in industrial applications during the preliminary design phases when the final rigidity and damping characteristics are not well known. Future applications may have variable flexibility characteristics during different operating stages, and these can also benefit from the criterion (when building large solar arrays or stations in space, for example). On the other hand, if the criterion is not satisfied, instability conditions can be determined in terms of general structural flexibility parameters.

II. Controller Model

As shown in Fig. 1, the controller comprises three elements in series: a sensor to measure the attitude error, θ; a compensator which converts the sensor output into a sutiable command; and the third element, the actuator. Sensors vary widely in accuracy and in the physical principle used to generate an estimate of attitude error. For greatest generality, we wish to confine attention to simple models; noting that a stability analysis will be performed, the most essential characteristic of a sensor in the present context is its tendency to introduce a time lag, τ_s. This is represented in Fig. 1 by $\omega_s(=1/\tau_s)$ and the sensor gain is denoted by k_s. Standard control-design considerations suggest the form of compensation shown in Fig. 1 – proportional-plus-derivative-plus-integral (PDI) feedback. The associated gains (k_P, k_D, k_I) are allowed only to be positive and will be selected in Sec. IV.

We wish also to model the actuator in a manner that is at once typical, yet simple. The simple transfer function shown in Fig. 1 meets these criteria. It may be thought to represent, for example, a reaction wheel. The fundamental dynamical principle of a reaction wheel is that when the system is subjected to an angular impulse, $\int T dt$, its angular momentum must increase by precisely $\int T dt$. Thus, in the case of external torques, the "system" consists of the spacecraft-plus-wheel,

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*Professor, Institute for Aerospace Studies.

†Control Systems Engineer.

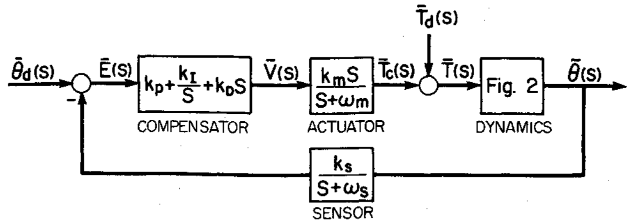

Fig. 1 Single-axis control system of a flexible spacecraft.

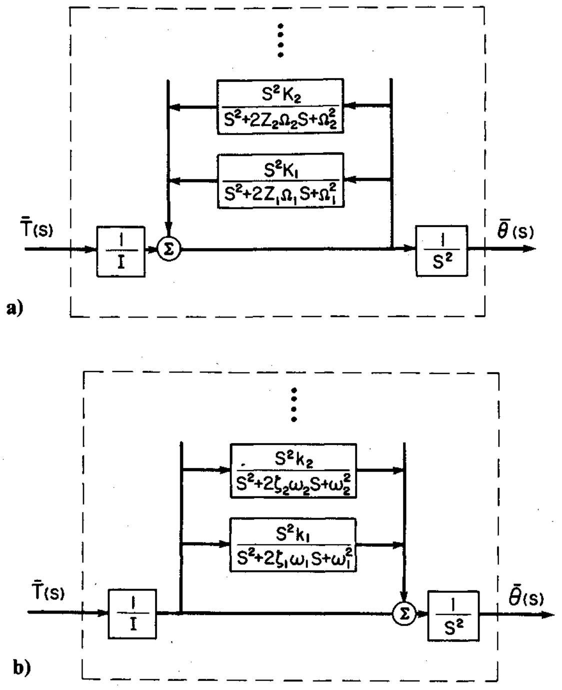

Fig. 2 Dynamic model of flexible spacecraft in terms of a) appendage modes and b) spacecraft modes.

and the controller insures that the angular momentum increase is passed on to the wheel, not the spacecraft. For the most common type of motor/wheel combinations, the transfer function between the voltage into the motor and the torque applied to the wheel (equal and opposite to the control torque thereby applied to the spacecraft) is as shown in Fig. 1. The term k_m denotes the actuator gain and ω_m is a characteristic frequency.

III. Selection of Controller Gains Based on Rigid-Body Poles

The simplest application of mode separability is to neglect all vibration modes and select the controller gains as though the spacecraft were rigid. The dynamical model in this case reduces to the transfer function $1/Is^2$, a limiting case of the models shown in Figs. 2a and 2b. The characteristic equation reduces to

$$s^4 + (\omega_s + \omega_m)s^3 + (\omega_s\omega_m + k_D')s^2 + k_P's + k_I' = 0 \qquad (1)$$

where

$$\begin{bmatrix} k_P' \\ k_D' \\ k_I' \end{bmatrix} = \frac{k_s k_m}{I} \begin{bmatrix} k_P \\ k_D \\ k_I \end{bmatrix} \qquad (2)$$

The selection of compensator gains would, in practice, be an involved process based on many performance criteria. Stability is a basic consideration. The nontrivial Routh-

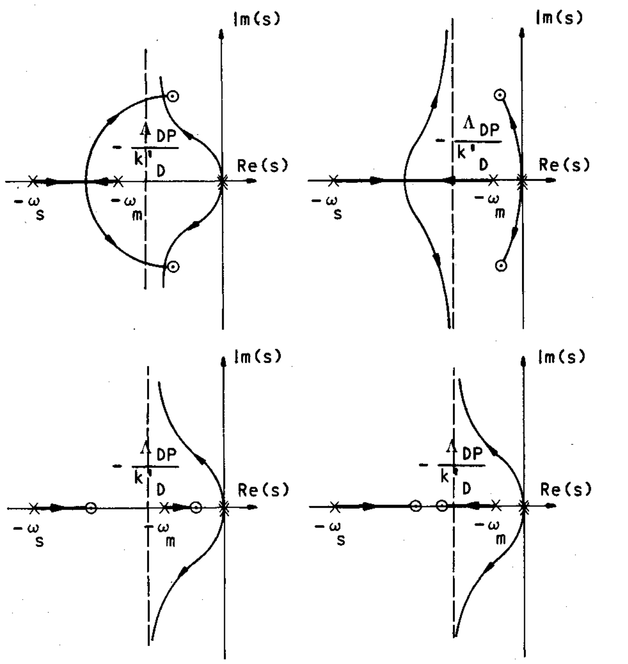

Fig. 3 Typical root loci.

Hurwitz criteria associated with the quartic equation (1) may be expressed as follows:

$$\Lambda_{DP} > -\omega_s\omega_m(\omega_s + \omega_m) \qquad (3)$$

$$(\omega_s + \omega_m)\Lambda_{PI} + k_P'\Lambda_{DP} > 0 \qquad (4)$$

where

$$\Lambda_{DP} \triangleq (\omega_s + \omega_m)k_D' - k_P'$$
$$\Lambda_{PI} \triangleq \omega_s\omega_m k_P' - (\omega_s + \omega_m)k'_I \qquad (5)$$

Also instructive is a look at the root loci (Fig. 3) for the rigid controller as k_D' varies (k_P'/k_D' and k_I'/k_D' are kept constant). From Fig. 1, there are two poles and two zeros in the left half-plane, and two poles at the origin. Two of the loci terminate in zeros, while two others tend to an asymptote that is parallel to the imaginary axis and that intersects the negative real axis provided

$$\Lambda_{DP} > 0 \qquad (6)$$

This condition applies for all k_D', k_P', k_I' as they are varied [in fixed proportion over $(0, \infty)$], and is therefore a stronger condition than Eq. (3). It can be shown from Eqs. (3), (4), and (6) that, provided

$$\Lambda_{DP} > 0 \qquad \Lambda_{PI} > 0 \qquad (7)$$

the root loci lie entirely in the left half-plane. This will be termed "unconditional stability." Yet another interpretation of these conditions is in terms of the phase-crossover frequency, ω_c. It can be shown to be given by

$$\omega_c^2 = -\Lambda_{PI}/\Lambda_{DP} \qquad (8)$$

If ω_c is real, the system is conditionally stable. On the other hand, if ω_c is imaginary (i.e., does not exist), the system is unconditionally stable.

To simplify matters and give a concrete example, a simple design strategy is illustrated as follows. One may wish to choose that the roots of the characteristic equation be moved as far into the left half s-plane as possible. More precisely, the gains (k'_P, k'_D, k'_I) are chosen such that the real part of the right-most root is (algebraically) minimized. This criterion has been discussed at some length in the literature[9,10] and leads, in the particular case of Eq. (1), to four roots with equal real parts:

$$s=-\sigma_r \pm j\sigma_r\sqrt{1-\zeta_{r1}^2}/\zeta_{r1} \qquad s=-\sigma_r \pm j\sigma_r\sqrt{1-\zeta_{r2}^2}/\zeta_{r2} \tag{9}$$

where ζ_{r1} and ζ_{r2} are the damping ratios corresponding to the rigid-body poles, and

$$\sigma_r=(\omega_s+\omega_m)/4 \tag{10}$$

To separate the rigid-body poles of Eq. (9) from the vibration-mode poles (the latter being near the imaginary axis), it might be reasonable to choose $\zeta_{r1}=\zeta_{r2}=1$. Unfortunately this choice leads to rather sluggish transient response, and the compromise in practice is to choose ζ_{r1} and ζ_{r2} in the range [0.5, 1.0].

From the characteristic equation (1) it follows that the rigid-body poles can be placed at Eq. (9) if the control gains are selected as follows:

$$\begin{aligned} k'_D &= \sigma_r^2(4+\zeta_{r1}^{-2}+\zeta_{r2}^{-2})-\omega_s\omega_m \\ k'_P &= 2\sigma_r^3(\zeta_{r1}^{-2}+\zeta_{r2}^{-2}) \\ k'_I &= \sigma_r^4\zeta_{r1}^{-2}\zeta_{r2}^{-2} \end{aligned} \tag{11}$$

where $\zeta_{r1}\le 1$ and $\zeta_{r2}\le 1$. Alternatively, the gains could be used to place the rigid-body poles at

$$s=-\sigma_r \pm \sigma_r\sqrt{\zeta_{r1}^2-1}/\zeta_{r1} \qquad s=-\sigma_r \pm \sigma_r\sqrt{\zeta_{r2}^2-1}/\zeta_{r2} \tag{12}$$

Equations (11) still apply although now $\zeta_{r1}\ge 1$ and $\zeta_{r2}\ge 1$, and the optimization criterion of placing the poles as far into the left half s-plane as possible has been abandoned. A combination of two pole pairs [one from Eq. (9) and one from Eq. (12)] can also be obtained by the same formulas.

IV. Structural Flexibility

It is assumed that the satellite consists of a relatively rigid main body to which are attached one or more flexible appendages, and that there is at least one axis about which the vehicle's attitude motion is uncoupled from the motion about the other two axes, and from translational motion. Let the (small) rotation about this axis be θ and the moment of inertia about this axis be denoted I. The motion equations then take the form[11]

$$\begin{aligned} I\ddot{\theta}+\Sigma h_n\ddot{q}_n &= T(t) \\ h_n\ddot{\theta}+I_f(\ddot{q}_n+\Omega_n^2 q_n) &= 0 \qquad (n=1,2,\ldots) \end{aligned} \tag{13}$$

where q_n is a "modal coordinate" associated with nth vibration mode of the appendages (i.e., with $\theta\equiv 0$; see the second equation above). The natural frequency of this mode is Ω_n, and its contribution to the angular momentum about the axis of rotation is $h_n\dot{q}_n$. I_f is the portion of I that is due to the flexible appendages and is used to normalize the mode shapes. As shown in Fig. 1, the satellite total torque $T(t)$ is the sum of control torque $T_c(t)$ and disturbance torque $T_d(t)$.

To control the attitude by means of appropriate torques, we desire a transfer function between the torque, T, and the attitude θ. Taking the Laplace transform of Eq. (13), we find that

$$s^2 I_e(s)\bar{\theta}(s)=\bar{T}(s) \tag{14}$$

where the inertance, $I_e(s)$, is given by

$$I_e(s)=I\left(1-\sum\frac{s^2K_n}{s^2+\Omega_n^2}\right) \tag{15}$$

and

$$K_n=h_n^2/II_f \tag{16}$$

In both Eqs. (13) and (15), the summation is taken over as many modes as are retained in the model. The block diagram for Eq. (15) is shown in Fig. 2a.

The model, Eq. (13), does not account for sources of energy dissipation in the structure. Although these dissipative effects are quite small, they can be crucial in a stability investigation; a small linear viscous damping term is therefore heuristically added for each mode. The denominator $s^2+\Omega_n^2$ in Eq. (15) is replaced by $s^2+2Z_n\Omega_n s+\Omega_n^2$ where $Z_n>0$. For normal structural damping $0<Z_n\ll 1$.

In the transfer function from torque to angle, it is actually the *inverse* of the inertance that is directly required. Anticipating our needs later, we define (see also Fig. 2b)

$$g(s)=\frac{I}{s^2 I_e(s)}=\frac{1}{s^2}+\sum\frac{k_n}{s^2+2\zeta_n\omega_n s+\omega_n^2} \tag{17}$$

where the parameters (k_n, ω_n, ζ_n) are determined from the appendage parameters (K_n, Ω_n, Z_n), and correspond to the "modes" of the spacecraft as a whole. The conversion from (K_n, Ω_n, Z_n) to (k_n, ω_n, ζ_n) is discussed to some extent in Ref. 12. We only note here that $k_n>0$, $\omega_n>0$ for all n. Also, $\zeta_n>0$ for all n unless all the $Z_n=0$ (which is physically unrealistic). This can be proved by setting $I_e(\sigma+j\omega)=0$ and deducing from the real and imaginary parts that $\sigma<0$ unless all $Z_n=0$. Since the zeros of $I_e(s)$ are the poles of $g(s)$, we conclude that $\zeta_n>0$ for real spacecraft.

We now know both the zeros and the poles of $g(s)$. This is more clearly evident in the alternative form

$$g(s)=\frac{\left(1+\sum_{n=1}^{N}k_n\right)\prod_{n=1}^{N}(s^2+2Z_n\Omega_n s+\Omega_n^2)}{s^2\prod_{n=1}^{N}(s^2+2\zeta_n\omega_n s+\omega_n^2)} \tag{18}$$

The constant multiplier is derived by comparing $s^2g(s)$ from Eqs. (17) and (18) and letting $s\to\infty$.

V. Effect of Flexibility on Stability

In the early days of satellites, attitude control gains were selected assuming the satellite to be rigid [for example, choosing gains according to Eq. (11)]. This occasionally led to compromised performance and even to failure for some missions.[13] This experience led to a policy of adjusting the "rigid-satellite" gains to account for flexibility.[14,15] On this basis, controllers could be designed over a range of flexibility parameters; outside this range, problems could still be encountered (usually with very flexible spacecraft; see, for example, Ref. 14). Moreover, stability is in many cases sensitive to the structural damping ratios, Z_n or ζ_n, which are often not determined accurately in practice.

We now derive stability conditions for the general situation where the spacecraft structure can be modeled as shown in Fig. 2b. The characteristic equation of the system (shown in Fig. 1) is

$$1+\frac{k'_I+k'_P s+k'_D s^2}{(s+\omega_s)(s+\omega_m)}g(s)=0 \tag{19}$$

where $g(s)$ is the transfer function described earlier by Eqs. (17) and (18).

Our intention is to use a root-locus argument. We therefore need the poles and zeros of the expression added to unity in Eq. (19). It is clear from Eqs. (18) and (19) that these poles and zeros are all in the left half-plane (except for the two poles at the origin). There are two more poles than zeros, and the two root loci that consequently tend to infinity can also be shown to have an asymptote parallel to the imaginary axis, and that lies farther in the left half-plane than if the vehicle were rigid.

The root loci, as any of k_P', k_D', k_I' is varied over the range $(0, \infty)$, thus begin and end in the left half-plane. To enter the right half-plane at some intermediate point on the locus, there must be a solution $s=j\omega$ Eq. (19) for real ω. Substituting $s=j\omega$ into Eq. (19), we arrive at the form

$$1+(\gamma+j\delta)(\alpha-j\beta)/\Delta=0 \tag{20}$$

where

$$\alpha = -\frac{1}{\omega^2}+\sum\frac{k_n(\omega_n^2-\omega^2)}{\Delta_n} \tag{21}$$

$$\beta = \sum\frac{2\zeta_n\omega_n\omega}{\Delta_n} \tag{22}$$

$$\gamma = (\omega_s\omega_m-\omega^2)(k_I'-k_D'\omega^2)+k_P'\omega^2(\omega_s+\omega_m) \tag{23}$$

$$\delta = \Lambda_{DP}\omega^3+\Lambda_{PI}\omega \tag{24}$$

$$\Delta = (\omega_s\omega_m-\omega^2)^2+(\omega_s+\omega_m)^2 \tag{25}$$

$$\Delta_n = (\omega_n^2-\omega^2)^2+2\zeta_n^2\omega_n^2\omega^2 \tag{26}$$

The real and imaginary parts of Eq. (20) are

$$\Delta+\alpha\gamma+\beta\delta=0 \tag{27}$$

$$\alpha\delta-\beta\gamma=0 \tag{28}$$

Substituting for α from Eq. (28) into Eq. (27), we must satisfy

$$\Delta+\beta(\gamma^2+\delta^2)/\delta=0 \tag{29}$$

Since Δ is clearly >0 from Eq. (25) and $\gamma^2+\delta^2\geq 0$, we must have $\beta/\delta<0$. For real structures, $\zeta_n>0$; consequently from Eq. (22), $\beta>0$. The only possibility, then, is $\delta<0$. If we can eliminate this possibility, we shall have prevented the root loci from ever crossing the imaginary axis into the right half-plane. It is therefore sufficient (although not necessary) that

$$\Lambda_{DP}>0 \qquad \Lambda_{PI}>0 \tag{30}$$

This guarantees that $\delta>0$ always, and that the system is stable for all flexibility parameters.

The conditions of Eq. (30) are valid for any design strategy involving k_P', k_D', and k_I'. If, in particular, we choose to use the strategy implied by Eq. (11), then Eq. (30) becomes

$$8+\zeta_{rI}^{-2}+\zeta_{r2}^{-2}>8/R \tag{31}$$

$$\zeta_{rI}^2+\zeta_{r2}^2>R/2 \tag{32}$$

where R is the squared ratio of arithmetic mean to geometric mean of ω_s and ω_m, that is,

$$R\triangleq(\omega_s+\omega_m)^2/4\omega_s\omega_m \tag{33}$$

It is evident that $1\leq R<\infty$, and that Eq. (31) is always satisfied. The condition of Eq. (32) is therefore sufficient to guarantee stability for any flexible spacecraft. It should be remembered, however, that this last condition is predicated not only on the PDI controller of Fig. 1, but also on the design strategy typified by Eqs. (9) and (12). The latter is a reasonable one having, as we have seen, excellent stability characteristics and leaving two parameters, ζ_{rI} and ζ_{r2}, free for further design improvement. If a different approach is used with a PDI controller, resort may still be had to the more basic stability conditions (30). More generally, even if one has a linear controller that is not PDI in form, an approach similar to the one taken in this section may still be productive.

Comparing the conditions of Eq. (30) with the earlier conditions of Eq. (7) we see that they are identical. This is expressed in the following theorem: "If a rigid-body PDI attitude controller is stable unconditionally with respect to gain, then it is also stable unconditionally with respect to structural flexibility." It is reiterated that the conditions of Eq. (30) are sufficient, but not necessary, for particular flexible spacecraft. This is illustrated in the following section. In some cases, gains that do not satisfy Eq. (30) may even improve controller performance.

VI. Numerical Results

In numerical work, it is of course necessary to choose specific values for (K_n, Ω_n, Z_n), and hence for (k_n, ω_n, ζ_n). To do this with as few assumptions as possible, we now choose three parameters and relate $(K_n, \Omega_n, Z_n, n=1,2,...)$ to them by means of plausible assumptions. The three parameters are: Ω_I, a measure of the degree of structural flexibility; I_f/I, a measure of the relative size of the appendages; and Z_I, which measures structural damping. Having thus specified Ω_I, we assume that the higher frequencies are bounded as follows:

$$n\leq\Omega_n/\Omega_I\leq n^2 \tag{34}$$

The bounds in Eq. (34) may not always be satisfied by all structures; it is nevertheless a reasonable model since $\Omega_n=n\Omega_I$ is satisfied by strings and membranes, and $\Omega_n=n^2\Omega_I$ is satisfied (as $n\to\infty$) by rods. It is similarly reasonable[12] to assume that

$$K_n\Omega_n^2=\text{constant} \tag{35}$$

To determine the value of this constant, we further assume a set of modes that are complete in the mathematical sense whence, from the theory of orthogonal functions,

$$\sum_{n=1}^{\infty}K_n=I_f/I \tag{36}$$

So for membrane-like behavior $(\Omega_n=n\Omega_I)$, it follows from Eqs. (34-36) that

$$K_n=6(I_f/I)/n^2\pi^2 \tag{37}$$

while for rod-like behavior $(\Omega_n=n^2\Omega_I)$,

$$K_n=90(I_f/I)/n^4\pi^4 \tag{38}$$

Just as Ω_I is a measure of appendage flexibility (and all the higher frequencies related to it), so also I_f/I is a measure of the relative "size" of the appendage (with all the modal "gains" K_n related to I_f/I). Lastly, with respect to the damping factors, Z_n, one assumes simply that

$$Z_I=Z_2=Z_3=\ldots \tag{39}$$

Thus the appendage is completely represented by only three independent parameters: Ω_I, I_f/I, and Z_I.

Stability results for rod-like and membrane-like appendages are found to be qualitatively equivalent. Small differences

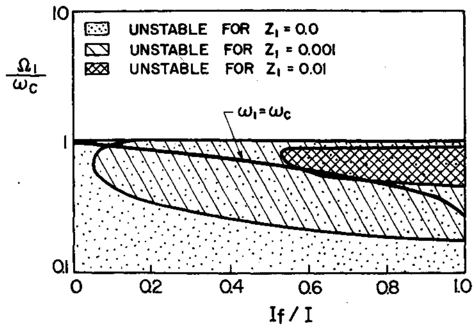

Fig. 4 Stability regions for three values of damping factor and "rod-like" behavior.

exist only near stability boundaries for $\Omega_I/\omega_c \leq 1$, when $Z_I = 0.001$ and 0.01. Some typical results are shown in Fig. 4, a stability plot for a rod-like appendage with $Z_I = 0.0$, 0.001, and 0.01 when $\omega_s = 0.5$ rad/s and $\omega_m = 0.26 \times 10^{-3}$ rad/s (the characteristic frequency of a wheel on a recent spacecraft, Hermes). More detailed information can be obtained with the aid of root loci (see Ref. 14). In interpreting Fig. 4, it should be remembered that although Ω_I, Ω_2, ... are the natural frequencies of the appendage (with the rigid main body constrained to be fixed), the frequencies of the overall spacecraft are the ones which cause $I_e(j\omega) = 0$ in Eq. (14), i.e., ω_I, ω_2,.... Formulas may be found in Ref. 12 for their calculation. As a one-mode approximation,

$$\omega_I \cong \Omega_I / \sqrt{I - K_I} \tag{40}$$

The locus of points for which $\omega_I = \omega_c$ is also shown in Fig. 4, and it is clear that "ω_c in the neighborhood of ω_I" is a more accurate description of the unstable region than "ω_c in the neighborhood of Ω_I."

If the sensor's characteristic frequency ω_s were changed for the conditionally stable cases, small variations would be observed in the unstable regions of Fig. 4. (Note that ω_c is also affected by ω_s.) On the other hand, if ω_s is fixed and either ζ_{r1} or ζ_{r2} is decreased, the region of instability expands slightly but remains below $\Omega_I/\omega_c = 1$; the regions remain quantitatively similar to those in Fig. 4. For all conditionally stable cases studied, the following rule of thumb avoids instability:

$$\Omega_I > \omega_c \tag{41}$$

Dissipative mechanisms in the appendage also were significant, however, and Eq. (41) appears to be necessary only as $Z_I \to 0$. No instabilities are observed for any (Ω_I/ω_c), (I_f/I), or ω_s, when $Z_I > 0.05$, for either rod-like and membrane-like appendages.

For the conditionally stable cases, it is interesting to note that the region $\Omega_I \ll \omega_c$ is stable when $Z_I > 0$. This may again be interpreted in terms of mode separability. The region is not likely to be an attractive one in practice, however, because large structural deformations can be expected there. Confirming the analysis, no instability was observed for the unconditionally stable cases when the same structural flexibility parameters of Fig. 4 were used with $Z_I = 0.0$, 0.001, and 0.01.

VII. Concluding Remarks

A new stability condition has been derived for the attitude control of flexible spacecraft. The feedback law is proportional-plus-derivative-plus-integral, the sensor and actuators have simple but representative models, and the structural flexibility has been essentially unrestricted. The stability condition guarantees that flexibility cannot destabilize the system if, when assumed rigid, the vehicle is unconditionally stable with respect to controller gains. It would be interesting to discover whether this condition remains true for a broader class of attitude controller.

The condition just mentioned is sufficient but not necessary. This was illustrated by the numerical examples, which showed that the system may be either stable or unstable if the condition is violated. It is likely to be unstable if the structural damping is very low and the characteristic frequency of the controller, ω_c, is close to the natural frequency of spacecraft vibration, ω_I. Not one case of instability was discerned for damping factors $Z_I > 5\%$; this underlines the benefits of adequate structural damping.

Acknowledgments

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